

# REPRESENTATIONS OF LOOP KAC-MOODY LIE ALGEBRAS

S. ESWARA RAO AND VYACHESLAV FUTORNY

**ABSTRACT.** We study representations of the Loop Kac-Moody Lie algebra  $\mathfrak{g} \otimes A$ , where  $\mathfrak{g}$  is any Kac-Moody algebra and  $A$  is a ring of Laurent polynomials in  $n$  commuting variables. In particular, we study representations with finite dimensional weight spaces and their graded versions. When we specialize  $\mathfrak{g}$  to be a finite dimensional or affine Lie algebra we obtain modules for toroidal Lie algebras.

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## INTRODUCTION

Let  $\mathfrak{g}_0$  be a simple finite dimensional Lie algebra and  $A$  the ring of Laurent polynomials in  $n$  commuting variables. The paper grew out of our interest to construct new modules for toroidal Lie algebras, which are the universal central extensions of  $\mathfrak{g}_0 \otimes A$ . In particular,  $\mathfrak{g}_0 \otimes A$  is a quotient of the corresponding toroidal Lie algebra and, hence, any module of  $\mathfrak{g}_0 \otimes A$  lifts to a module of this toroidal Lie algebra. Let  $\mathfrak{g}_{\text{aff}}$  be the the affinization of  $\mathfrak{g}_0$ . Then  $\mathfrak{g}_{\text{aff}} \otimes A$  is a quotient of a toroidal Lie algebra with  $n+1$  variables (see [E2], Example 4.2). Thus, by constructing modules for  $\mathfrak{g}_0 \otimes A$  and  $\mathfrak{g}_{\text{aff}} \otimes A$ , we obtain modules for corresponding toroidal Lie algebras. In fact, it is shown in [E3] that any irreducible module over a toroidal Lie algebra with finite dimensional weight spaces comes from some module over  $\mathfrak{g}_0 \otimes A$  or  $\mathfrak{g}_{\text{aff}} \otimes A$ .

Let  $\mathfrak{g}$  be any Kac-Moody Lie algebra,  $\mathfrak{h} \subseteq \mathfrak{g}$  a Cartan subalgebra. In this paper we study highest weight modules for  $\mathfrak{g} \otimes A$  for any Kac- Moody Lie algebra  $\mathfrak{g}$  as our methods work in such generality. The main idea comes from the paper of Wilson [W], where the author studies the case of  $\mathfrak{g} = s\ell_2$  and  $A = \mathbb{C}[t, t^{-1}]$ . We generalize most of the results of [W] for any Kac-Moody Lie algebra and for any number of variables.

Set  $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$  and  $\mathfrak{h}' = \mathfrak{h} \cap \mathfrak{g}'$ . Let  $\mathfrak{g}' = N^+ \oplus \mathfrak{h}' \oplus N^-$  be the standard decomposition. Then  $\mathfrak{g}' \otimes A = (N^+ \otimes A) \oplus (\mathfrak{h}' \otimes A) \oplus (N^- \otimes A)$  is a natural triangular decomposition. We study irreducible modules  $V$  for  $\mathfrak{g}' \otimes A$  such that  $V$  admits a nonzero element  $v$  such that  $(N^+ \otimes A)v = 0$  and  $\mathfrak{h}' \otimes A$  acts on  $v$  via some function  $\psi \in (\mathfrak{h}' \otimes A)^*$ . We will denote the corresponding module by  $V(\psi)$ . Note that  $V(\psi)$  is a weight module, that is

$$V(\psi) = \bigoplus_{\mu \in (\mathfrak{h}')^*} V(\psi)_\mu.$$

Module  $V(\psi)$  will be called a *highest weight* module with a highest weight  $\psi$  (though the construction is similar to the construction of loop modules in the affine case).

We are mainly interested in modules  $V(\psi)$  having finite dimensional weight spaces  $V(\psi)_\mu$  for all  $\mu \in (\mathfrak{h}')^*$ . Necessary and sufficient conditions for a module  $V(\psi)$  to have all finite dimensional weight spaces were given in [E2]: there must exist a co-finite ideal  $I$  of  $A$  such that

$$(\mathfrak{g}' \otimes I)V(\psi) = 0.$$

Billig and Zhao defined exp-polynomial maps (see Definition 3.1) and showed that  $V(\psi)$  has finite dimensional weight spaces if  $\psi$  is an exp-polynomial map [BZ]. Rencai Lu and Zhao proved that this condition is necessary [RZ]. They also gave an explicit formula for the exp-polynomial map in terms of the co-finite ideal  $I$ . This explicit form is very useful for us in the study of  $V(\psi)$ .

Note that  $\mathfrak{g}' \otimes A$  is naturally  $\mathbb{Z}^n$ -graded with a gradation coming from  $A$ . Hence, it is natural and important to study  $\mathbb{Z}^n$ -graded modules for  $\mathfrak{g}' \otimes A$ . Though module  $V(\psi)$  is not  $\mathbb{Z}^n$ -graded, we can define a natural  $\mathfrak{g}' \otimes A$ -module structure on  $V(\psi) \otimes A$ , which is already  $\mathbb{Z}^n$ -graded. Even though  $V(\psi)$  is irreducible, the module  $V(\psi) \otimes A$  need not be irreducible. It was shown in [E2] that  $V(\psi) \otimes A$  is completely reducible with finitely many irreducible components. The interplay between modules  $V(\psi)$  and irreducible components of  $V(\psi) \otimes A$  is the main content of this paper.

Given any function  $\psi \in (\mathfrak{h}' \otimes A)^*$  one can define a natural  $\mathbb{Z}^n$ -graded map  $\overline{\psi}$  from  $\mathfrak{h}' \otimes A$  to  $A$  and a  $\mathbb{Z}^n$ -graded irreducible module  $V(\overline{\psi})$ . This module is an irreducible component of  $V(\psi) \otimes A$  ([E2]). Moreover, all irreducible components are graded isomorphic up to a grade shift.

Suppose  $V(\psi) \otimes A$  decomposes into  $R$  isomorphic copies. Then the map  $\psi$  has an interesting decomposition takes.  $\psi$  can be decomposed into a sum of  $R$  exp-polynomial maps (see Proposition 3.4). In addition, each of these exp-polynomial maps gives rise to an irreducible highest weight module. Further  $V(\psi)$  is isomorphic to a tensor product of these modules (see Corollary 3.9). Moreover, all these components of the tensor product are isomorphic as  $\mathfrak{g}' \otimes A$ -modules up to an automorphism of  $\mathfrak{g}' \otimes A$  (see Section 5). On the other hand, the restriction of any of these automorphisms to  $\mathfrak{g}'$  is identity. Hence, all components of the decomposition of  $V(\psi)$  into the tensor product are isomorphic as  $\mathfrak{g}'$ -modules. In particular, they are isomorphic as  $\mathfrak{h}'$ -modules. So the character of  $V(\psi)$  is the product of characters of one of these components (see Proposition 5.1).

We recall an abstract decomposition of  $V(\psi) \otimes A$  from an earlier work of the first author in Section 1. One of our main results in Theorem 4.2 where we give an explicit description of the components of this decomposition in terms of automorphisms discussed earlier. This allows us to compute the characters of the components of  $V(\psi) \otimes A$  in terms of the character of  $V(\psi)$ . In Section 5 we discuss in detail the case  $n = 2$ .

We also note that Theorem 3.4 of [PB] can be easily deduced from our Proposition 3.4. The polynomials  $P_{h,\lambda}$  are in fact constant in this case.

## 1. PRELIMINARIES

All vector spaces, algebras and tensor products are over complex numbers  $\mathbb{C}$ . Let  $\mathbb{Z}, \mathbb{N}$  and  $\mathbb{Z}_+$  denote integers, non-negative integers and positive integers, respectively. For any Lie algebra  $\mathcal{A}$  denote  $U(\mathcal{A})$  the universal enveloping algebra of  $\mathcal{A}$ .

Recall that  $\mathfrak{g}$  is an arbitrary Kac-Moody Lie-algebra (see [K] for details) and  $A = \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$  is the ring of Laurent polynomials in  $n$  commuting variables. Write  $\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{h}''$  for some subspace  $\mathfrak{h}''$  of  $\mathfrak{h}$ .

Set

$$\begin{aligned}\mathfrak{g}_A &= \mathfrak{g}' \otimes A \oplus \mathfrak{h}'' \supseteq \mathfrak{g} \\ \mathfrak{h}_A &= \mathfrak{h}' \otimes A \oplus \mathfrak{h}'' \supseteq \mathfrak{h}.\end{aligned}$$

For  $m = (m_1, \dots, m_n) \in \mathbb{Z}^n$  denote  $t^m = t_1^{m_1} \dots t_n^{m_n} \in A$ . The Lie bracket in  $\mathfrak{g}_A$  is defined as follows:

$$\begin{aligned}[X \otimes t^m, Y \otimes t^s] &= [X, Y] \otimes t^{m+s} \\ [h, X \otimes t^m] &= [h, X] \otimes t^m \\ [h, h'] &= 0 \text{ for}\end{aligned}$$

$X, Y \in \mathfrak{g}', h, h' \in \mathfrak{h}'', m, s \in \mathbb{Z}^n$ . The Lie algebra  $\mathfrak{g}_A$  is called *Loop Kac-Moody Lie algebra*.

Let  $D$  be  $\mathbb{C}$ -linear span of derivations  $d_1, \dots, d_n$  where  $[d_i, \mathfrak{g}' \otimes t^m] = m_i \mathfrak{g}' \otimes t^m$  and  $[d_i, d_j] = [d_i, \mathfrak{h}''] = 0$  for all  $i, j = 1, \dots, n$ .

Denote  $\widetilde{\mathfrak{g}}_A = \mathfrak{g}_A \oplus D, \widetilde{\mathfrak{h}}_A = \mathfrak{h}_A \oplus D$ . Let  $\mathfrak{g} = N^- \oplus \mathfrak{h} \oplus N^+$  be the standard triangular decomposition. Then  $\mathfrak{g}_A = (N^- \otimes A) \oplus \mathfrak{h}_A \oplus (N^+ \otimes A)$  is a triangular decomposition of  $\mathfrak{g}_A$ .

Let  $\psi : \mathfrak{h}_A \rightarrow \mathbb{C}$  be any linear map. Define the induced map  $\overline{\psi} : \mathfrak{h}_A \rightarrow \mathbb{C}$  as follows:

$$\overline{\psi}(h \otimes t^m) = \psi(h \otimes t^m) t^m$$

for  $h \in \mathfrak{h}', m \in \mathbb{Z}^n$ , and  $\overline{\psi}(\mathfrak{h}'') = \psi(\mathfrak{h}'')$ . Let  $\mathbb{C}$  be the one dimensional representation of  $(N^+ \otimes A) \oplus \mathfrak{h}_A$  where  $N^+ \otimes A$  acts trivially and  $\mathfrak{h}_A$  acts via  $\psi$ :  $h \cdot 1 = \psi(h)1$  for any  $h \in \mathfrak{h}_A$ . Consider the induced module for  $\mathfrak{g}_A$ ,

$$M(\psi) = U(\mathfrak{g}_A) \bigotimes_{(N^+ \otimes A) \oplus \mathfrak{h}_A} \mathbb{C}.$$

By standard arguments one can show that  $M(\psi)$  has a unique irreducible quotient denoted by  $V(\psi)$ .

Now we will define a graded version of this module. We consider  $A$  as a  $\widetilde{\mathfrak{h}}_A$ -module with respect to the following action:

$$\begin{aligned}(h \otimes t^m) \cdot t^k &= \overline{\psi}(h \otimes t^m) t^k \\ h' \cdot t^m &= \overline{\psi}(h') t^m \\ d_i \cdot t^m &= m_i t^m.\end{aligned}$$

for  $h \in \mathfrak{h}'$ ,  $h' \in \mathfrak{h}''$ ,  $d_i \in D$ ,  $i = 1, \dots, n$ ,  $m, k \in \mathbb{Z}^n$ . Extend  $\bar{\psi}$  to  $U(\mathfrak{h}' \otimes A)$  by an algebra homomorphism. Denote the image  $\bar{\psi}(U(\mathfrak{h}' \otimes A))$  by  $A_\psi$ , which is also a  $\tilde{\mathfrak{h}}_A$ -module.

We have the following criteria of irreducibility.

**Lemma 1.1.** *[E1], Lemma 1.2] The  $\tilde{\mathfrak{h}}_A$ -module  $A_\psi$  is an irreducible if and only if each homogeneous element of  $A_\psi$  is invertible in  $A_\psi$ .*

Throughout this paper we assume that  $A_\psi$  is an irreducible  $\tilde{\mathfrak{h}}_A$ -module. Define

$$\text{Supp } \bar{\psi} = \{m \in \mathbb{Z}^n \mid (A_\psi)_m \neq 0\}.$$

Clearly,  $\text{Supp } \bar{\psi}$  is a subgroup of  $\mathbb{Z}^n$ .

Suppose now that  $N^+ \otimes A$  acts trivially on  $A_\psi$ . Consider the induced module for  $\tilde{\mathfrak{g}}_A$ ,

$$M(\bar{\psi}) = U(\tilde{\mathfrak{g}}_A) \bigotimes_{(N^+ \otimes A) \oplus \tilde{\mathfrak{h}}_A} A_\psi.$$

Since  $A_\psi$  is an irreducible module, it follows that  $M(\bar{\psi})$  has a unique irreducible quotient  $V(\bar{\psi})$ . It is standard that  $M(\psi)$  and  $M(\bar{\psi})$  are weight module with respect to  $\mathfrak{h}$  and  $\mathfrak{h} \oplus D$ , respectively, that is

$$M(\psi) = \bigoplus_{\mu \in \mathfrak{h}^*} M(\psi)_\mu$$

and

$$M(\bar{\psi}) = \bigoplus_{\mu \in (\mathfrak{h} \oplus D)^*} M(\bar{\psi})_\mu$$

(We refer to [E1], Section 3 for detail). Since any quotient of a weight module is a weight module,  $V(\psi)$  and  $V(\bar{\psi})$  are irreducible weight modules for  $\mathfrak{g}_A$  and  $\tilde{\mathfrak{g}}_A$ , respectively.

Our first goal is to establish a relationship between  $V(\bar{\psi})$  and  $V(\psi)$ . We define now  $\tilde{\mathfrak{g}}_A$  module structure on  $V(\psi) \otimes A$  as follows:

$$\begin{aligned} (X \otimes t^m) \cdot (v \otimes t^k) &= ((X \otimes t^m) \cdot v) \otimes t^{m+k} \\ h \cdot (v \otimes t^k) &= (hv) \otimes t^k \\ d_i \cdot (v \otimes t^k) &= k_i(v \otimes t^k) \end{aligned}$$

for  $v \in V(\psi)$ ,  $X \in \mathfrak{g}'$ ,  $h \in \mathfrak{h}''$ ,  $d_i \in D$ ,  $i = 1, \dots, n$ ,  $m, k \in \mathbb{Z}^n$ . We have the following

**Proposition 1.2.** *[E2], Proposition 3.5] Let  $G \subset \mathbb{Z}^n$  be such that  $\{t^m, m \in G\}$  is a set of coset representatives for  $A/A_\psi$ . Let  $v$  be a highest weight vector in  $V(\psi)$ . Set  $v(m) = v \otimes t^m$  for any  $m \in \mathbb{Z}^n$ . Then*

- (1)  $V(\psi) \otimes A = \bigoplus_{m \in G} Uv(m)$  where  $Uv(m)$  is the  $\tilde{\mathfrak{g}}_A$ -submodule generated by  $v(m)$ .
- (2) Each  $Uv(m)$  is an irreducible  $\tilde{\mathfrak{g}}_A$ -module.
- (3)  $Uv(0) \cong V(\bar{\psi})$  as  $\tilde{\mathfrak{g}}_A$ -modules.
- (4) All components in (1) are isomorphic as  $\tilde{\mathfrak{g}}_A$ -modules up to a grade shift, that is the  $D$  action is shifted by a vector in  $\mathbb{C}^n$ .

We now recall some standard results on  $M(\psi)$  and  $M(\bar{\psi})$ . We first observe that  $\tilde{\mathfrak{g}}_A$  is a Pre- exp-polynomial algebra in the sense of [[BGLZ], Example 1].

- Lemma 1.3.** (1) *[BGLZ], Theorem 2.12] The  $\tilde{\mathfrak{g}}_A$ -module  $M(\bar{\psi})$  is irreducible if and only if  $M(\psi)$  is an irreducible  $\mathfrak{g}_A$ -module.*
- (2) *[E2], Lemma 3.6] The  $\tilde{\mathfrak{g}}_A$  module  $V(\bar{\psi})$  has finite dimensional weight spaces with respect to  $\mathfrak{h} \oplus D$  if and only if  $\mathfrak{g}_A$  module  $V(\psi)$  has finite dimensional weight spaces with respect to  $\mathfrak{h}$ .*
- (3) *[E2], Lemma 3.7] The  $\mathfrak{g}_A$  module  $V(\psi)$  has finite dimensional weight spaces with respect to  $\mathfrak{h}$  if and only if  $\psi$  factors through  $\mathfrak{h}' \otimes A/I$  for some co-finite ideal  $I$  of  $A$ .*

**Remark 1.** *From the Proof of Lemma 3.7 in [E2], we can choose the co-finite ideal  $I$  to be generated by polynomials  $P_i$  in the variable  $t_i$  with nonzero constant term, see also [BGLZ], Theorem 2.9.*

## 2. MODULES WITH FINITE DIMENSIONAL SPACES

In this section we discuss modules with finite dimensional weight spaces. We have seen in Section 1 that  $V(\psi)$  has finite dimensional weight spaces if there exist polynomials  $P_1, \dots, P_n$  in variable  $t_1, \dots, t_n$  which generate a co-finite ideal  $I$ . We can

assume that each polynomial is not a constant. Indeed, otherwise the ideal  $I$  coincides with the algebra  $A$ ,  $\psi$  is a trivial function and the corresponding module is one dimensional.

Set  $\Gamma = \text{Supp } \overline{\psi}$ . Under our assumption  $\Gamma$  is a subgroup of  $\mathbb{Z}^n$ . (see below Lemma 1.1)

**Lemma 2.1.** *The rank of  $\Gamma$  equals  $n$ .*

*Proof.* Suppose that the rank of  $\Gamma$  is less than  $n$ . After a change of variables we can assume that

$$h \otimes t^m \cdot t_n^k v = 0,$$

for all  $k \in \mathbb{Z} \setminus \{0\}$  and  $m \in \mathbb{Z}^n$  such that  $m_n = 0$ . On the other hand, there exists a non-constant polynomial  $P_n$  in variable  $t_n$  such that  $h \otimes t^m \cdot P_n(t_n)v = 0$ . Now it is easy to conclude that  $(h \otimes t^m) \cdot v = 0$  for all  $m \in \mathbb{Z}^n$ . Thus the function  $\psi$  is trivial and  $V(\psi)$  is a one dimensional module, which is of no interest.  $\square$

It follows from Lemma 2.1 that

$$A_\psi = \mathbb{C}[t_1^{\pm r_1}, \dots, t_n^{\pm r_n}]$$

and

$$\Gamma = r_1 \mathbb{Z} \oplus \dots \oplus r_n \mathbb{Z}$$

for some positive integers  $r_1, \dots, r_n$ . Set  $R = r_1 r_2 \dots r_n$ . If  $R = 1$  then  $V(\psi) \otimes A \cong V(\overline{\psi})$  as  $\tilde{\mathfrak{g}}_A$ -modules.

From now on we assume that  $R \geq 2$ . Then  $V(\psi) \otimes A$  decomposes into  $R$  irreducible  $\tilde{\mathfrak{g}}_A$ -modules by Proposition 1.2(1). We will give a more explicit description of these components in Section 4.

**Proposition 2.2.** *For each  $i, 1 \leq i \leq n$ , there exists a  $\mathfrak{g}$ -module automorphism  $\sigma_i$  of  $V(\psi)$  of order  $r_i$ .*

*Proof.* Fix  $i \in \{1, \dots, n\}$ . Let  $\xi_i$  be the  $r_i$ -th primitive root of unity. Define an algebra automorphism  $\sigma_i$  of  $A$  as follows:  $\sigma_i(t_i) = \xi_i t_i$  and  $\sigma_i(t_j) = t_j$  for  $i \neq j$ . It can be extended to an automorphism of  $\mathfrak{g}' \otimes A$  by the identity action on  $\mathfrak{g}'$ . Also, defining

the identity action of  $\sigma_i$  on  $h''$  we obtain an automorphism of order  $r_i$  on  $\mathfrak{g}_A$ . It can be further extended to an algebra automorphism of  $U(\mathfrak{g}_A)$ . This automorphism, clearly, respects the triangular decomposition of  $\mathfrak{g}_A$ . Let  $J$  be the left ideal generated by  $N^+ \otimes A$ ,  $h \otimes t^m - \psi(h \otimes t^m)$  and  $h' - \psi(h')$ ,  $h \in \mathfrak{h}'$ ,  $h' \in \mathfrak{h}''$  and  $m \in \mathbb{Z}^n$ .

Suppose that  $m_i \not\equiv 0(r_i)$ . Then we have

$$\begin{aligned} & \sigma_i(h \otimes t^m - \psi(h \otimes t^m)) \\ &= h \otimes \xi_i^{m_i} t^m \\ &= \xi_i^{m_i} (h \otimes t^m - \psi(h \otimes t^m)). \end{aligned}$$

On the other hand, if  $m_i \equiv 0(r_i)$  then we have

$$\begin{aligned} & \sigma_i(h \otimes t^m - \psi(h \otimes t^m)) \\ &= h \otimes \xi_i^{m_i} t^m - \psi(h \otimes t^m) \\ &= h \otimes t^m - \psi(h \otimes t^m), \end{aligned}$$

as  $\xi_i^{m_i} = 1$ . This implies that  $\sigma_i$  leaves the ideal  $J$  invariant. Since  $M(\psi)$  can be identified with  $U(\mathfrak{g}_A)/J$ , we obtain that  $\sigma_i$  induces an automorphism of  $M(\psi)$ .

Suppose now that  $N$  is a proper submodule of  $M(\psi)$ . Let  $\sigma_i((X_{\beta_1} \otimes t^{m^1}) \cdots (X_{\beta_\ell} \otimes t^{m^\ell})v) \in \sigma_i(N)$ , where  $X_{\beta_j} \in \mathfrak{g}'$ ,  $m^1, \dots, m^\ell \in \mathbb{Z}^n$ . Then we have

$$\begin{aligned} & (X_\beta \otimes t^m) \cdot \sigma_i((X_{\beta_1} \otimes t^{m^1}) \cdots (X_{\beta_\ell} \otimes t^{m^\ell})v) \\ &= \xi_i^{-m_i} \sigma_i((X_\beta \otimes t^m)(X_{\beta_1} \otimes t^{m^1}) \cdots (X_{\beta_\ell} \otimes t^{m^\ell})v) \in \sigma_i(N). \end{aligned}$$

Thus  $\sigma_i(N)$  is also a submodule of  $M(\psi)$ . Clearly,  $\sigma_i(N)$  is a proper submodule, as  $\sigma_i(v) = v$  for a highest weight vector  $v$ . We conclude that the sum of all proper submodules  $M(\psi)$  is invariant under  $\sigma_i$ . Hence,  $\sigma_i$  is an automorphism on  $V(\psi)$  of order  $r_i$ .  $\square$

Let  $\bar{\Gamma} = \mathbb{Z}/_{r_1\mathbb{Z}} \oplus \cdots \oplus \mathbb{Z}/_{r_n\mathbb{Z}} \cong \mathbb{Z}^n/\Gamma$ . We have the following immediate corollary.

**Corollary 2.3.** (1) *For any  $k = (k_1, \dots, k_n) \in \bar{\Gamma}$  there exists a  $\mathfrak{g}$ -module automorphism  $\eta_k = \sigma_1^{k_1} \cdots \sigma_n^{k_n}$  of  $V(\psi)$  of finite order.*

(2) *For any  $k = (k_1, \dots, k_n) \in \bar{\Gamma}$  there exists a finite order automorphism  $\tau_k$  of  $\tilde{\mathfrak{g}}_A$  such that  $\tau_k(X \otimes t^m) = \xi_1^{k_1 m^1} \cdots \xi_n^{k_n m^n} X \otimes t^m$  for  $X \in \mathfrak{g}'$ ,  $m = (m_1, \dots, m_n) \in \mathbb{Z}^n$  and  $\tau_k(\mathfrak{g} \oplus D) = Id$ .*



### 3. IRREDUCIBLE COMPONENTS OF $V(\psi) \otimes A$

In this section we give a description of the components of  $V(\psi) \otimes A$  as a  $\tilde{\mathfrak{g}}_A$  module in terms of the automorphisms constructed in Corollary 2.3.

First, recall the definition of exp-polynomial maps from [BZ].

**Definition 3.1.** *A function  $f : \mathbb{Z}^n \rightarrow \mathbb{C}$  is called an exp-polynomial map if  $f$  can be written as a finite sum*

$$f(m_1, \dots, m_n) = \sum_{\substack{a \in (\mathbb{C}^*)^n \\ k \in \mathbb{Z}^n}} C_{k,a} m_1^{k_1} \dots m_n^{k_n} a_1^{m_1} \dots a_n^{m_n},$$

$C_{k,a} \in \mathbb{C}$  and  $a = (a_1, \dots, a_n) \in (\mathbb{C}^*)^n, k = (k_1, \dots, k_n)$

Given  $h \in \mathfrak{h}'$  define the function  $\psi_h : \mathbb{Z}^n \rightarrow \mathbb{C}$  as follows:

$$\psi_h(m_1, \dots, m_n) := \psi(h \otimes t_1^{m_1} \dots t_n^{m_n})$$

for  $m_1, \dots, m_n \in \mathbb{Z}$ .

**Lemma 3.2.** *The following conditions are equivalent for  $\psi \in (\mathfrak{h}_A)^*$ .*

- (1) *There exist polynomials  $P_1, \dots, P_n$  in variables  $t_1, \dots, t_n$  such that*
- (R)  *$\psi(\mathfrak{h}' \otimes AP_i(t_i)) = 0$*
- (2)  *$\psi_h$  is an exp-polynomial map for all  $h \in \mathfrak{h}'$ .*
- (3)  *$V(\psi)$  has finite dimensional weight spaces with respect to  $\mathfrak{h}$ .*

*Proof.* Fix  $h \in \mathfrak{h}'$  and consider  $\psi_h$ . Taking  $\psi_h$  in Lemma 2.7 of [RZ], (1)  $\Leftrightarrow$  (2) follows. Just note that (2.4) in Lemma 2.7 of [RZ] is precisely our relation (R). Also note that the expression 2.5 in Lemma 2.7 of [RZ] is an exp-polynomial map.

Now 2  $\Leftrightarrow$  3 follows from Lemma 1.3(3) and Remark 1.  $\square$

Given  $a = (a_1, \dots, a_n) \in (\mathbb{C}^*)^n$ , define the exp-polynomial function  $f_a : \mathbb{Z}^n \rightarrow \mathbb{C}$  as follows:

$$f_a(m_1, \dots, m_n) = m_1^{k_1} \dots m_n^{k_n} a_1^{m_1} \dots a_n^{m_n}.$$

**Lemma 3.3.** *Functions  $f_a$  are linearly independent for different  $a \in (\mathbb{C}^*)^n$ .*

*Proof.* Follows from [[BZ], Corollary 2.4].  $\square$

Given  $\lambda = (\lambda_1, \dots, \lambda_n)$  define  $\exp \lambda : \mathbb{Z}^n \rightarrow \mathbb{C}$  by

$$\exp \lambda(m_1, \dots, m_n) = \lambda_1^{m_1} \cdots \lambda_n^{m_n}.$$

Assume that the image of  $\overline{\psi}$  equals  $A_\psi = \mathbb{C}[t_1^{\pm r_1} \cdots, t_n^{\pm r_n}]$  and  $R = r_1 r_2 \cdots, r_n$ .

Define  $\overline{\Gamma}$  action on  $(\mathbb{C}^*)^n$  as  $k \cdot \lambda = (\xi_1^{k_1} \lambda_1, \dots, \xi_n^{k_n} \lambda_n)$  where  $k = (k_1, \dots, k_n) \in \overline{\Gamma}$  and  $\xi_i$  is the  $r_i$ -th primitive root of unity. Write  $\xi^k = (\xi_1^{k_1}, \dots, \xi_n^{k_n})$ .

Let  $B$  be a set of coset representatives of this action.

Define operators  $T_k$  such that

$$T_k \cdot \exp(\lambda) = \exp(k \cdot \lambda) \text{ for } k \in \overline{\Gamma}.$$

Further let  $P_R = \sum_{k \in \overline{\Gamma}} T_k$ .

**Proposition 3.4.** *Suppose  $\psi : \mathfrak{h}' \otimes A \rightarrow \mathbb{C}$  is an exp-polynomial map, that is the map  $\psi_h(m_1, \dots, m_n) = \psi(h \otimes t^m)$  is an exp-polynomial for any  $h \in \mathfrak{h}'$ . Then we have*

$$\psi_h = P_R \sum_{\lambda \in B} p_{h,\lambda} \exp \lambda,$$

with  $p_{h,\lambda}$  being some polynomial function of the form:

$$p_{h,\lambda}(m_1, \dots, m_n) = \sum_{\ell \in \mathbb{Z}^n} C_{h,\lambda,\ell} m_1^{\ell_1} \cdots m_n^{\ell_n},$$

where  $\lambda \in B$ ,  $h \in \mathfrak{h}'$ , the sum is finite and  $C_{h,\lambda,\ell}$  are constants depending on  $h, \lambda$  and  $\ell$ .

*Proof.* For each  $h \in \mathfrak{h}'$  write the function  $\psi_h$  in the form:

$$\psi_h(m_1, \dots, m_n) = \sum_{\lambda \in (\mathbb{C}^*)^n} p_{h,\lambda}(m_1, \dots, m_n) \lambda_1^{m_1} \cdots \lambda_n^{m_n},$$

where  $p_{h,\lambda}$  is some polynomial map for every  $\lambda \in (\mathbb{C}^*)^n$ . Then for a fixed  $k = (k_1, \dots, k_n) \in \overline{\Gamma}$  we have

$$\psi_h(m_1, \dots, m_n) = \sum_{\lambda \in (\mathbb{C}^*)^n} P_{h,k,\lambda} (\xi_1^{k_1} \lambda_1)^{m_1} \cdots (\xi_n^{k_n} \lambda_n)^{m_n}.$$

Since  $\psi_h(m_1, \dots, m_n) = 0$  if  $m_i \not\equiv 0(r_i)$  for some  $i$ , and since  $(\xi_j^{k_j})^{r_j} = 1$  for all  $j$ , we have

$$\psi_h(m_1, \dots, m_n) = \sum_{\lambda \in (\mathbb{C}^*)} p_{h,k,\lambda} \lambda_1^{m_1} \dots \lambda_n^{m_n}.$$

By Lemma 3.3, we have

$$p_{h,\lambda} = p_{h,k,\lambda} \text{ for all } k \in \bar{\Gamma}.$$

Thus if  $\lambda$  occurs in the summation then  $\xi^k \lambda$  also occurs with the same coefficient. Hence, we have

$$\psi_h = P_R \sum_{\lambda \in B} p_{h,\lambda} \exp \lambda.$$

Note that in this summation if  $\lambda$  occurs then  $k.\lambda$  does not occur (this is the meaning of  $B$ ). Also note that such an expression for  $\psi_h$  need not be unique.  $\square$

We would now like to prove that  $V(\psi)$  admits a certain tensor product decomposition if  $\psi$  is a exp-polynomial map. First we will prove certain results on exp-polynomial maps. We need to use Lemma 2.7 of [RZ].

Let  $\psi$  be an exp-polynomial map,  $P_1, \dots, P_n$  some polynomials in  $t_1, \dots, t_n$ , respectively, such that

$$\psi(h \otimes AP_i(t_i)) = 0,$$

for all  $h \in \mathfrak{h}', i = 1, \dots, n$ .

We can assume that the leading terms of these polynomials equal 1. Then we have

$$\psi_h = \sum_{\lambda \in (\mathbb{C}^*)^n} p_{h,\lambda} \exp \lambda.$$

For each  $i = 1, \dots, n$ , the  $i$ -th degree of  $p_{h,\lambda}(m_1, \dots, m_n)$  be the maximal degree of  $m_i$ . This degree does not depend on  $h$  and only depends on the polynomial  $P_i$ .

Suppose  $\lambda = (\lambda_1, \dots, \lambda_n)$  occurs in the summation above. Then  $\lambda_i$  is a root of  $P_i$  for each  $i$  by (2.5) in Lemma 2.7 of [RZ]. The multiplicity of  $\lambda_i$  is the  $i$ -th degree of  $p_{h,\lambda}$  plus one. Further, we have seen in the proof of Proposition 3.4 that if  $\lambda_i$  occurs in the sum then  $\xi_i^{k_i} \lambda_i$  also occurs with the same coefficient. Hence, if  $\lambda_i$  is a root of  $P_i$  then  $\xi_i^{k_i} \lambda_i$  is also a root of  $P_i$  with the same multiplicity. Then we can write

$$P_i(t_i) = \prod_{j=1}^{s_i} \prod_{\ell=1}^{r_i} (t_i - \xi_i^\ell a_{ij})^{b_{ij}},$$

for some scalars  $a_{ij}, b_{ij}$  and  $s_i$  depending on  $P_i$ . Further,  $(a_{ij}/a_{ij'})^{r_i} = 1$  implies  $j = j'$ . Define

$$P_{i,\ell}(t_i) = \prod_{j=1}^{k_i} (t_i - \xi_i^\ell a_{ij})^{b_{ij}}.$$

For each  $i = 1, \dots, n$  fix  $j_i$ ,  $1 \leq j_i \leq r_i$ . Set  $J = (j_1, \dots, j_n)$  and consider the ideal  $I_J$  of  $A$  generated by  $P_{1,j_1} \cdots P_{n,j_n}$ .

**Lemma 3.5.** *Let  $J' = (j'_1, \dots, j'_n)$ ,  $1 \leq j'_i \leq r_i$ ,  $i = 1, \dots, n$ . Suppose  $J \neq J'$ . Then the ideals  $I_J$  and  $I_{J'}$  are co-prime, that is  $I_J + I_{J'} = A$ .*

*Proof.* Since  $J \neq J'$  there exists  $i$ ,  $1 \leq i \leq n$ , such that  $j_i \neq j'_i$ . Then the polynomial  $P_{i,j_i}$  and  $P_{i,j'_i}$  have no common roots. Thus the ideal of  $\mathbb{C}[t_i, t_i^{-1}]$  generated by  $P_{i,j_i}$  and  $P_{i,j'_i}$  coincides with  $\mathbb{C}[t_i, t_i^{-1}]$ , which implies the statement.  $\square$

Chinese Remainder theorem implies immediately the following statement.

**Proposition 3.6.** *Let*

$$I = \prod_{J \in \overline{\Gamma}} I_J = \bigcap_{J \in \overline{\Gamma}} I_J.$$

*Then*

$$\mathfrak{g}' \otimes A/I \simeq \mathfrak{g}' \otimes \left( \bigoplus_{J \in \overline{\Gamma}} A/I_J \right).$$

We also have

**Lemma 3.7.** *[E2], Remark 3.9] Let  $\psi : \mathfrak{h}_A \rightarrow \mathbb{C}$  be a linear map satisfying  $\psi(\mathfrak{h}' \otimes I) = 0$  for some co-finite ideal  $I$  of  $A$ . Then  $(\mathfrak{g}' \otimes I)V(\psi) = 0$ .*

Let  $\alpha_1, \dots, \alpha_k$  be a set of simple roots of  $\mathfrak{g}$ . We will consider a standard ordering on  $\mathfrak{h}^*$ : for  $\eta_1, \eta_2 \in \mathfrak{h}^*$  we say that  $\eta_1 \leq \eta_2$  if and only if  $\eta_2 - \eta_1 = \sum_i n_i \alpha_i$  for some non-negative integers  $n_i$ 's.

**Proposition 3.8.** *Let  $I_1$  and  $I_2$  be co-prime co-finite ideals of  $A$ . Let  $\psi_1$  and  $\psi_2$  be linear maps from  $\mathfrak{h}_A \rightarrow \mathbb{C}$  such that  $\psi_i(\mathfrak{h}' \otimes I_i) = 0$  for  $i = 1, 2$ . Then*

$$V(\psi_1 + \psi_2) \simeq V(\psi_1) \otimes V(\psi_2)$$

*as  $\mathfrak{g}_A$ -modules.*

*Proof.* We will show first that  $V(\psi_1) \otimes V(\psi_2)$  is a cyclic module generated by  $v_1 \otimes v_2$  where  $v_i$  is a highest weight vector of  $V(\psi_i)$  for  $i = 1, 2$ .

Since  $I_1$ , and  $I_2$  are co-prime, we have  $I_1 + I_2 = A$ . Thus, there exists  $f_i \in I_i$ ,  $i = 1, 2$ , such that  $f_1 + f_2 = 1$ . For  $X \in \mathfrak{g}'$  and  $h \in A$  consider

$$\begin{aligned} Xf_1h(v_1 \otimes v_2) &= v_1 \otimes Xf_1hv_2 \\ &= v_1 \otimes (Xh - Xf_2h)v_2 \\ &= v_1 \otimes Xhv_2, \end{aligned}$$

as  $Xf_1hv_1 = Xf_2hv_2 = 0$  by Lemma 3.7. Repeating this process we see that the module generated by  $v_1 \otimes v_2$  contains  $v_1 \otimes V(\psi_2)$ . Now taking  $Xf_2h$  instead of  $Xf_1h$  we obtain that the module generated by all elements  $v_1 \otimes w$ ,  $w \in V(\psi_2)$  contains  $V(\psi_1) \otimes V(\psi_2)$ . We conclude that  $v_1 \otimes v_2$  generates  $V(\psi_1) \otimes V(\psi_2)$ . Hence,  $V(\psi_1 + \psi_2)$  is a homomorphic image of  $V(\psi_1) \otimes V(\psi_2)$ . To complete the proof it is sufficient to show that  $V(\psi_1) \otimes V(\psi_2)$  is irreducible. Let

$$(1) \quad v = \sum_{\lambda + \mu = \eta, i} v_{\lambda i} \otimes v_{\mu i} \in V(\psi_1) \otimes V(\psi_2)$$

be a vector of weight  $\eta$ . We can assume that  $\{v_{\lambda i}\}$  is a linearly independent set. Note that  $\lambda$  and  $\mu$  may occur several times but with different vectors. That is why the additional index  $i$  is used. Now choose  $\mu'$  to be the minimal among the  $\mu$ 's that occur in (1) with respect to the ordering defined above. Fix a weight vector  $v_{\mu' s}$  of weight  $\mu'$  in (1). Then there exists  $X \in U(\mathfrak{g}_A)_{\psi_2 - \mu'}$  such that  $Xv_{\mu' s} = v_2$  and  $Xv_{\mu' i}$  is a multiple of  $v_2$  for all  $i$ . Using the arguments as above, we obtain that there exists  $X' \in U(\mathfrak{g}_A)$  such that  $X'v = \sum_{\lambda + \mu = \eta, i} v_{\lambda i} \otimes Xv_{\mu i}$ . We claim that  $w = Xv_{\mu i} = 0$  for  $\mu \neq \mu'$ . Indeed, if  $w \neq 0$  then it has the weight  $\psi_2 - \mu' + \mu \leq \psi_2$  implying that  $\mu < \mu'$ . But this contradicts the minimality of  $\mu'$ . Hence,  $Xv_{\mu i} = 0$  for  $\mu \neq \mu'$  and any  $i$ . Thus

$$X'v = \sum v_{(\eta - \mu')i} \otimes k_i v_2,$$

for some  $k_i \in \mathbb{C}$ . This element cannot be zero since  $v_{(\eta - \mu')i}$  is a linearly independent set and at least one term is nonzero. Thus we proved that given an element  $v$  in

$V(\psi_1) \otimes V(\psi_2)$  there exists  $X' \in U(\mathfrak{g}_A)$  such that  $X'v = w \otimes v_2$  for some nonzero  $w \in V(\psi_1)$ . Repeating this argument one easily shows that the module generated by  $w \otimes v_2$  contains  $v_1 \otimes v_2$ . This completes the proof.  $\square$

Let  $\psi$  be an exp-polynomial map. For any  $h \in \mathfrak{h}'$  set

$$\Phi_h = \sum_{\lambda \in B} p_{h,\lambda} \exp \lambda.$$

Then

$$\psi_h = P_R \sum_{\lambda \in B} p_{h,\lambda} \exp \lambda = \sum_{k \in \bar{\Gamma}} \xi^k \Phi_h.$$

Clearly,  $\xi^k \Phi_h$  is an exp-polynomial map for any  $k \in \bar{\Gamma}$ . We can choose  $\lambda$ 's in  $\Phi_h$  in such a way that corresponding polynomials are  $P_{1,r_1}, \dots, P_{n,r_n}$ . So the polynomials corresponding to the exp-polynomial map  $\xi^k \Phi_h$  are  $P_{1,k_1}, \dots, P_{n,k_n}$  where  $k = (k_1, \dots, k_n)$ . If  $k' = (k'_1, \dots, k'_n)$  then the ideals  $I_k$  and  $I_{k'}$  are mutually co-prime if  $k \neq k'$  by Lemma 3.5. Hence, we have the following:

**Corollary 3.9.**

$$V(\psi) \simeq \bigotimes_{k \in \bar{\Gamma}} V(\xi^k \Phi)$$

as  $\mathfrak{g}_A$ -modules.

We also have the following statement which is of independent interest.

**Lemma 3.10.**  $A_\Phi = \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ .

*Proof.* Clearly we have

$$A_\psi \subseteq A_\Phi = \mathbb{C}[t_1^{\pm s_1}, \dots, t_n^{\pm s_n}]$$

for some  $s_i \mid r_i$ . Write

$$\Phi_h = \sum_{\lambda \in B} p_{h,\lambda} \exp \lambda.$$

By Lemma 3.4 applied to  $\Phi_h$ , if  $\lambda$  occurs in the summation of  $\Phi_h$ , then  $\xi'_i \lambda$  also occurs where  $\xi'_i$  is the  $s_i$ th root of unity. As  $s_i \mid r_i$ ,  $\xi'_i$  is also  $r_i$  the root of unity. That is a contradiction to the definition of  $B$  if  $s_i > 1$ . Thus  $s_i = 1$  and the Lemma is proved.  $\square$

4. EXPLICIT DESCRIPTION OF COMPONENTS OF  $V(\psi) \otimes A$ 

In this section we describe explicitly the irreducible components of  $V(\psi) \otimes A$ . Recall that  $V(\psi) \otimes A$  is a  $\tilde{\mathfrak{g}}_A$ -module (see Section 1). Also recall that for each  $k \in \bar{\Gamma}$  there exists an automorphism  $\eta_k$  of  $V(\psi)$  of finite order by Corollary 2.3(1).

For  $k \in \bar{\Gamma}$  define the map

$$\tilde{\eta}_k : V(\psi) \otimes A \rightarrow V(\psi) \otimes A$$

induced by  $\eta_k$  as follows:

$$\tilde{\eta}_k(v \otimes p(t)) = \eta_k(v) \otimes p(\xi^{-k}t),$$

for all  $v \in V(\psi)$  and  $p(t) \in A$ .

**Lemma 4.1.** *The map  $\tilde{\eta}_k$  is an automorphism of  $V(\psi) \otimes A$  as a  $\tilde{\mathfrak{g}}_A$ -module.*

*Proof.* We need to check only that  $\tilde{\eta}_k$  is a  $\tilde{\mathfrak{g}}_A$ -module homomorphism. For  $X \in \mathfrak{g}'$ ,  $m \in \mathbb{Z}^n$ ,  $p(t) \in A$  and  $v \in V(\psi)$  we have

$$\begin{aligned} & \tilde{\eta}_k((X \otimes t^m)(v \otimes p(t))) \\ &= \tilde{\eta}_k(((X \otimes t^m)v) \otimes t^m p(t)) \\ &= \eta_k((X \otimes t^m)v) \otimes (\xi^{-k}t)^m p(\xi^{-k}t) \\ &= (X \otimes (\xi^k)^m t^m) \eta_k(v) \otimes \xi^{-km} t^m p(\xi^{-k}t) \\ &= (X \otimes t^m) \eta_k(v) \otimes t^m p(\xi^{-k}t). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} & (X \otimes t^m) \tilde{\eta}_k(v \otimes p(t)) \\ &= (X \otimes t^m)(\eta_k(v) \otimes p(\xi^{-k}t)) \\ &= (X \otimes t^m) \eta_k(v) \otimes t^m p(\xi^{-k}t). \end{aligned}$$

We conclude that  $\tilde{\eta}_k$  is a  $\tilde{\mathfrak{g}}_A$ -module map, which completes the proof.  $\square$

For any  $k = (k_1, \dots, k_n) \in \bar{\Gamma}$  set

$$(V(\psi) \otimes A)_k = \{v \in V(\psi) \otimes A \mid \tilde{\eta}_i v = \xi_i^{k_i} v, 1 \leq i \leq n\}.$$

It follows immediately from Lemma 4.1 that  $(V(\psi) \otimes A)_k$  is a  $\tilde{\mathfrak{g}}_A$ -module.

**Theorem 4.2.** *Let  $v$  be a highest weight vector of  $V(\psi)$ . Then*

$$U(\widetilde{\mathfrak{g}}_A)(v \otimes t^k) = (V \otimes A)_k,$$

for all  $k \in \overline{\Gamma}$ .

*Proof.* Note that  $v \otimes 1 \in (V(\psi) \otimes A)_0$  and  $v \otimes t^k \in (V(\psi) \otimes A)_k$  for  $k \in \overline{\Gamma}$ . Thus

$$U(\widetilde{\mathfrak{g}}_A)(v \otimes t^k) \subseteq (V(\psi) \otimes A)_k.$$

We also have

$$\begin{aligned} V(\psi) \otimes A &= \bigoplus_{k \in \overline{\Gamma}} U(\widetilde{\mathfrak{g}}_A)v \otimes t^k \\ &\subseteq \bigoplus_{k \in \overline{\Gamma}} (V(\psi) \otimes A)_k, \end{aligned}$$

by Proposition 1.2. It implies immediately that

$$U(\widetilde{\mathfrak{g}}_A)(v \otimes t^k) = (V \otimes A)_k.$$

□

## 5. CHARACTERS

One of the main problems in representation theory is to compute the characters of irreducible modules. We will indicate some partial results in this direction for modules under consideration.

Let  $\psi$  be an exp-polynomial map. Following the previous section we write

$$\psi = \sum_{k \in \overline{\Gamma}} \xi^k \Phi,$$

where each  $\xi^k \Phi$  is an exp-polynomial map.

For each  $k \in \overline{\Gamma}$  we fix an automorphism of finite order  $\sigma_k$  of  $\mathfrak{g}_A$ , which exists by Corollary 2.3, (2). Let

$$\mathfrak{g}_A \xrightarrow{p_{\xi^k}} \text{End}(V(\xi^k \Phi))$$

be the representation map. Then, clearly,

$$p_{\xi^k} \circ \sigma_\ell = p_{\xi^{k+\ell}},$$

for all  $k, \ell \in \overline{\Gamma}$ .



Since  $\sigma_\ell$  is *id* on  $\mathfrak{g}$ , we obtain the following isomorphism of  $\mathfrak{g}$ -modules:

$$V(\xi^k \Phi) \simeq V(\xi^{k+\ell} \Phi).$$

In particular,

$$V(\xi^k \Phi) \stackrel{\in_k}{\simeq} V(\Phi), \quad \forall k \in \bar{\Gamma}.$$

Set

$$W = \bigotimes_{R\text{-times}} V(\Phi).$$

Then we have the following isomorphisms of  $\mathfrak{g}$ -modules:

$$V(\psi) \stackrel{\Omega}{\simeq} \bigotimes_{k \in \bar{\Gamma}} V(\xi^k \Phi) \stackrel{\otimes \in_k}{\simeq} W.$$

Let  $\gamma = \otimes \in_k \circ \Omega$ . It is easy to check that  $\gamma \circ \eta_k \circ \gamma^{-1} : W \rightarrow W$  is a  $\mathfrak{g}$ -module map ( $\eta_k$  is defined in Corollary 2.3, (1)).

Since

$$\xi^\ell \psi = \sum_{k \in \bar{\Gamma}} \xi^{k+\ell} \Phi,$$

for any  $\ell \in \bar{\Gamma}$ , we see that  $\eta_\ell$  induces the isomorphism

$$\bigotimes_{k \in \bar{\Gamma}} V(\xi^k \Phi) \simeq \bigotimes_{k \in \bar{\Gamma}} V(\xi^{k+\ell} \Phi).$$

Hence,  $\eta_\ell$  and  $\gamma \circ \eta_\ell \circ \gamma^{-1}$  permute the factors of the tensor product.

**Proposition 5.1.** *Let  $\text{ch } V(\psi)$  and  $\text{ch } V(\Phi)$  be the characters of  $V(\psi)$  and  $V(\Phi)$ , respectively. Then*

$$\text{ch } V(\psi) = (\text{ch } V(\Phi))^R.$$

*Proof.* Note that we have the following isomorphism of  $\mathfrak{g}$ -modules:

$$V(\psi) \simeq \bigotimes_{R\text{-times}} V(\Phi).$$

In particular, this is an isomorphism of  $\mathfrak{h}$ -modules. Hence, their characters are same and the statement follows.  $\square$

**Character of  $(V(\psi) \otimes A)_k$  :** Now we will indicate how to compute the character of  $(V(\psi) \otimes A)_k$ . Since all the components are isomorphic upto a grade shift, it is sufficient to compute the character of  $(V(\psi) \otimes A)_0$ . Recall that

$$(V(\psi) \otimes A)_0 = \{v \in V(\psi) \otimes A_0 \mid \tilde{\eta}_k v = v, \forall k \in \bar{\Gamma}\}.$$

Let  $k = (k_1, \dots, k_n) \in \bar{\Gamma}$  and  $\eta_k = \sigma_1^{k_1} \cdots \sigma_n^{k_n}$ . Hence, it is sufficient to describe fixed points of all  $\tilde{\sigma}_\ell$ . Set

$$V(\psi)_{(k_1, \dots, k_n)} = \{v \in V(\psi) \mid \sigma_i v = \xi_i^{k_i} v, 1 \leq i \leq n\}.$$

Then we have

$$(V(\psi) \otimes A)_0 = \bigoplus_{k=(k_1, \dots, k_n) \in \mathbb{Z}^n} V(\psi)_{(k_1, \dots, k_n)} \otimes t^k.$$

Therefore, to compute the character of  $(V(\psi) \otimes A)_0$  it is sufficient to compute the characters of  $V(\psi)_{(k_1, \dots, k_n)}$ ,  $k \in \bar{\Gamma}$ .

Let  $V$  be an  $\mathbb{N}^d$ -graded vector space for some positive integer  $d$ . Here  $\mathbb{N}$  denotes the non-negative integers and  $\mathbb{N}^d$  denotes  $d$  copies of  $\mathbb{N}$ .

Let  $V = \bigoplus_{\alpha \in \mathbb{N}^d} V(\alpha)$ , a direct sum of graded components. We assume that each  $\dim V(\alpha)$  is finite. Fix a positive integer  $r$ . Consider  $V^r = V \otimes \cdots \otimes V$  ( $r$  times). Then  $V^r = \bigoplus_{\alpha \in \mathbb{N}^d} V^r(\alpha)$ , and

$$V^r(\alpha) = \bigoplus_{\sum \alpha_i = \alpha} V(\alpha_1) \otimes \cdots \otimes V(\alpha_r).$$

Clearly,  $\dim V^r(\alpha) < \infty$ .

Define  $\sigma_r : V^r \rightarrow V^r$  as follows:

$$\sigma_r(v_1 \otimes \cdots \otimes v_r) = v_r \otimes v_1 \otimes \cdots \otimes v_{r-1},$$

for all  $v_1, \dots, v_r \in V$ .

We have that  $\sigma_r(V^r(\alpha)) = V^r(\alpha)$ , for any  $\alpha \in \mathbb{N}^d$ . For each  $k \in \mathbb{Z}$  denote

$$V_k^r = \{v \in V^r \mid \sigma_r(v) = \xi^k v\},$$

where  $\xi$  is a primitive  $r$ -th root of unity. Then

$$V^r = \bigoplus_{k \in \mathbb{Z}/r\mathbb{Z}} V_k^r,$$

where

$$V_k^r = \bigoplus_{\alpha \in \mathbb{N}^d} V_k^r(\alpha).$$

For any  $\mathbb{N}^d$ -graded vector space  $U = \bigoplus_{\alpha \in \mathbb{N}^d} U(\alpha)$  define

$$P_U(X) = \sum_{\alpha \in \mathbb{N}^d} \dim U(\alpha) X^\alpha \in \mathbb{N}[[X_1, \dots, X_d]]$$

where  $X^\alpha = X_1^{\alpha_1} \cdots X_d^{\alpha_d}$ .

Set

$$C_r(n) = \sum_{t|gcd(r,n)} t\mu(r/t),$$

where  $\mu$  is the Mobius function.

We now recall the following formula from [W]. It is proved in [W] only for  $d = 1$  but the same proof holds for any  $d$ .

**Theorem 5.2.** [ [W], Theorem 5.5]

$$P_{V_k^r}(X) = \frac{1}{r} \sum_{t|r} C_t(r) (P_V(X^t))^{\frac{r}{t}}.$$

We will now indicate how to compute the character for  $(V(\psi) \otimes A)_0$  for the case  $n = 2$  (hence,  $R = r_1 r_2$ ). General case can be treated similarly but it requires more complicated notation.

Recall that  $V(\psi)$  is a highest weight module for  $\mathfrak{g}_A$  and  $V(\psi) \simeq \bigotimes_{R\text{-times}} V(\Phi)$ . Then  $V(\Phi)$  is a highest weight module for  $\mathfrak{g}_A$ . Let  $d$  be the rank of the Kac-Moody Lie algebra  $\mathfrak{g}$ . Let  $\alpha_1, \dots, \alpha_d$  be the simple roots of  $\mathfrak{g}$ ,  $Q^+$  be the set consisting of all non-negative linear combinations of roots  $\alpha_1, \dots, \alpha_d$ . Then  $Q^+$  can be identified with  $\mathbb{N}^d$ . We have that

$$V(\Phi) = \bigoplus_{\alpha \in \mathbb{N}^d} V(\Phi)_{\lambda - \alpha},$$

where  $\Phi|_{\mathfrak{h}} = \lambda$ .

Denote  $V = V(\Phi)$  and  $V(\alpha) = V(\Phi)_{\lambda-\alpha}$ .

Let

$$\begin{aligned} W &= V \otimes \cdots, \otimes V \text{ (} r_1 \text{ times) and} \\ V(\psi) &= W \otimes \cdots, \otimes W \text{ (} r_2 \text{ times).} \end{aligned}$$

Note that each space  $V(\xi^k \Phi)$  is identified with  $V$  and  $\sigma_1, \sigma_2$  are permutations on  $V(\psi)$ . Now it is easy to see that the sum  $\sum \xi^k \Phi$  can be rearranged in such way that  $\sigma_1$  leaves each  $W$  invariant and

$$\sigma_1(v_1 \otimes \cdots, \otimes v_{r_1}) = v_{r_1} \otimes v_1 \otimes \cdots, \otimes v_{r_1-1},$$

for all  $v_i \in V$ . This extends to  $V(\psi)$  leaving each component  $W$  invariant. Further we can assume

$$\begin{aligned} &\sigma_2(w_1 \otimes \cdots, \otimes \cdots, w_{r_2}) \\ &= w_{r_2} \otimes w_1 \otimes \cdots, \otimes w_{r_2-1}, w_i \in W. \end{aligned}$$

Since

$$V_{k_1}^{r_1} = W_{k_1} = \{v \in V^{r_1} \mid \sigma_1(v) = \xi_1^{k_1} v\},$$

where  $\xi_1$  is a  $r_1$ -th primitive root of unity, then we have

$$(2) \quad P_{W_{k_1}}(X) = \frac{1}{r_1} \sum_{t \mid r_1} C_t(r_1) (P_V(X^t))^{r_1/t},$$

by Theorem 5.2.

Now we have that  $W = \bigoplus_{(\alpha, s) \in \mathbb{N}^{d+1}} W(\alpha)_s$ , where  $W(\alpha)_s$  is the  $s$ 's eigenspace of  $\sigma_1$ .

Note that  $W$  is graded by  $\mathbb{N}^{d+1}$ . Again by Theorem 5.2 we have:

$$(3) \quad P_{W_{k_2}^{r_2}}(X) = P_{V(\psi)_{k_2}}(X) = \frac{1}{r_2} \sum_{t \mid r_2} C_t(r_2) P_W(X^t)^{\frac{r_2}{t}}.$$

Formula (3) implies that  $\dim V(\psi)_{(k_1, k_2)}(\alpha)$  can be computed in terms of  $\dim W_{k_1}(\alpha)$ . On the other hand,  $\dim W_{k_1}(\alpha)$  can be computed in terms of  $\dim V(\alpha)$  by (2). Hence, if we know the character of  $V(\Phi)$  then we can compute the character of any  $V(\psi)_{(k_1, k_2)}$ . This in turn allow us to compute the character of  $(V(\psi) \otimes A)_0$ .

We refer to [G1] and [G2] for character formulas for graded integrable modules.

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SCHOOL OF MATHEMATICS, TATA INSTITUTE OF FUNDAMENTAL RESEARCH, HOMI BHABHA ROAD, COLABA, MUMBAI, 400 005, INDIA

*E-mail address:* `senapati@math.tifr.res.in`

INSTITUTO DE MATEMÁTICA E ESTATÍSTICA, UNIVERSIDADE DE SÃO PAULO, CAIXA POSTAL 66281, SÃO PAULO, CEP 05315-970, BRASIL

*E-mail address:* `futorny@ime.usp.br`